

# Atmospheric Dynamics

## Revision topics

Stephan Matthiesen

Institute for Atmospheric and Environmental Sciences

School of Geosciences, Edinburgh University

Crew Building room 218

# Main structure of course

- Expressing Newton's laws for a fluid on a rotating sphere
- Consequence of scale (Rossby Number) gives geostrophic and higher approximations for wind in terms of pressure (or geopotential)
- Variation of wind with height
  - In free atmosphere – “thermal” wind
  - In boundary layer, eddy friction, and Ekman spiral
- Circulations in the tropics (introduced via “dishpan” results)
- Notions of vorticity, divergence, and potential vorticity
- Use of those concepts in
  - Deepening (2-D and 3-D)

# Newton's Laws on sphere

Preliminaries: hydrostatic equation, potential temperature and static stability

Material Derivative

$$s = s(t, x, y, z) \quad \delta s = \frac{\partial s}{\partial t} \delta t + \frac{\partial s}{\partial x} \delta x + \frac{\partial s}{\partial y} \delta y + \frac{\partial s}{\partial z} \delta z$$

$$\frac{Ds}{Dt} = \left( \frac{\partial s}{\partial t} + u \frac{\partial s}{\partial x} + v \frac{\partial s}{\partial y} + w \frac{\partial s}{\partial z} \right)$$

Rate of change of momentum of lump = surface integral of Pressure forces + vol integral of gravity force.

Leads (after applying Gauss's theorem) to

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho} \nabla p + \mathbf{g}_t$$

# Effect of rotating frame

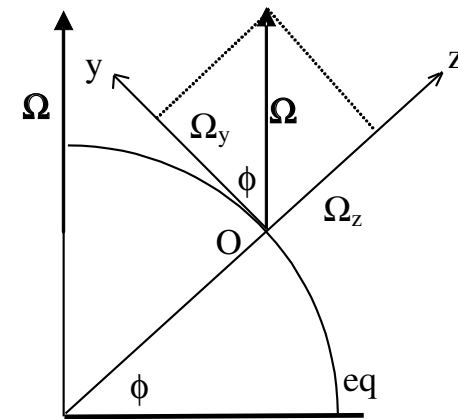
$$\left(\frac{D}{Dt}\right)_f = \left(\frac{D}{Dt}\right)_r + \boldsymbol{\Omega} \times \left(\frac{D}{Dt}\right) \mathbf{v} + 2\boldsymbol{\Omega} \times \mathbf{v} = -\frac{1}{\rho} \nabla p + g$$

$$\boldsymbol{\Omega} \wedge \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Omega_x & \Omega_y & \Omega_z \\ u & v & w \end{vmatrix}$$

$$\frac{Du}{Dt} + 2\Omega w \cos \phi - 2\Omega v \sin \phi = -\frac{1}{\rho} \frac{\partial p}{\partial x}$$

$$\frac{Dv}{Dt} + 0 + 2\Omega u \sin \phi = -\frac{1}{\rho} \frac{\partial p}{\partial y}$$

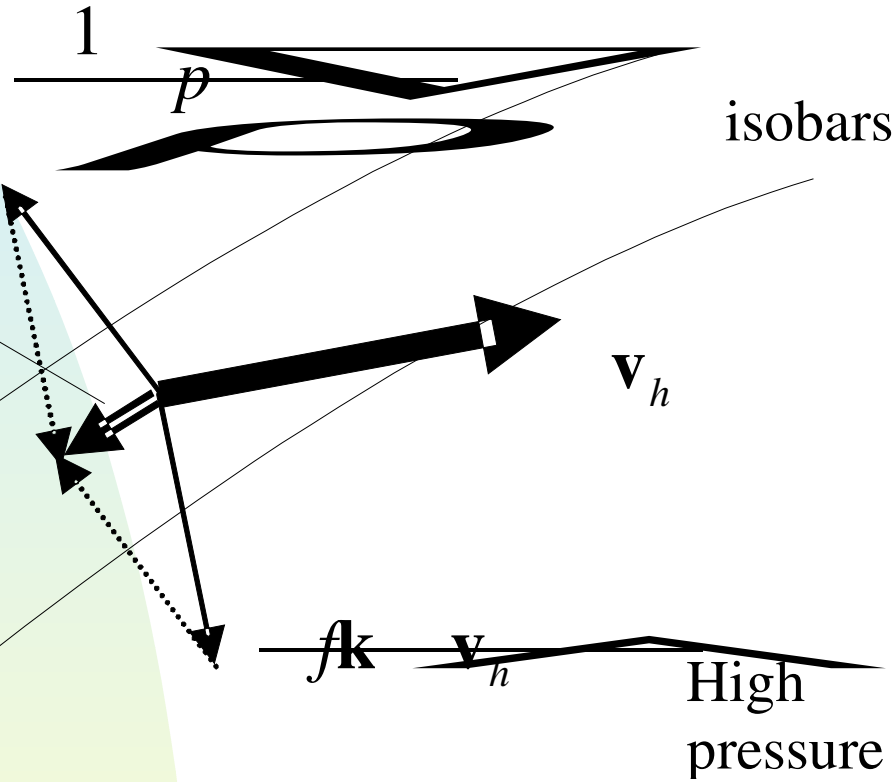
$$\frac{Dw}{Dt} + 0 - 2\Omega u \cos \phi = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g$$



# Summary of forces

low pressure

Accel



# Rossby No. and geostrophy

Rossby number,  $R_o$ , is the ratio of typical values of the relative acceleration and Coriolis acceleration

$$R_o = (U^2/L)/(fU) = U/(Lf)$$

$$U=10\text{ms}^{-1}, L=1000\text{km}, f\sim 10^{-4} \text{ gives } R_o=1/10$$

Define the geostrophic wind,  $\mathbf{v}_g$ , for which s.c.f. exactly balances s.p.g.f.

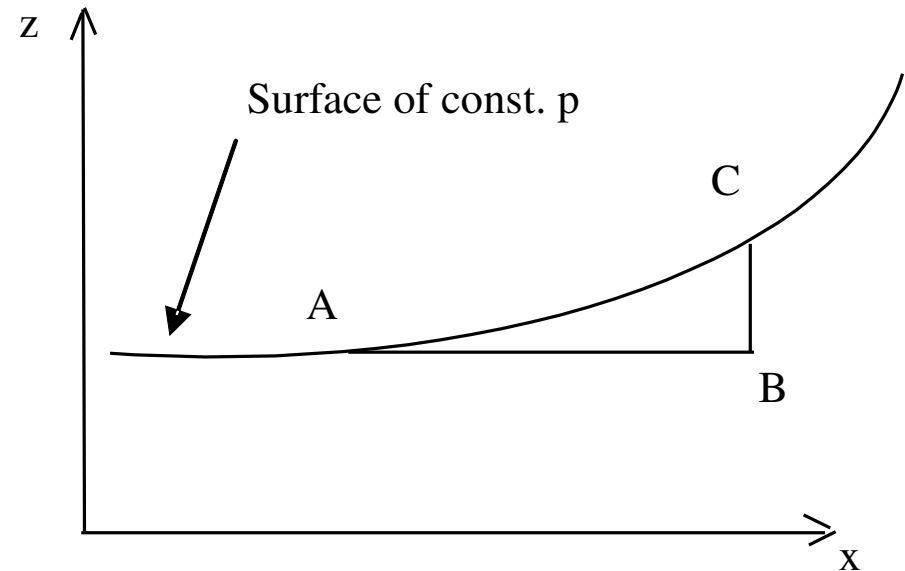
$$-f\mathbf{k} \wedge \mathbf{v}_g - \frac{1}{\rho} \nabla p \equiv 0$$

$$\mathbf{v}_g \equiv \frac{1}{\rho f} \mathbf{k} \wedge \nabla p$$

Higher approx to true wind can be found from the pressure field if acceleration can be estimated

- directly (as in gradient wind – steady circular flow)
- by using geostrophic approximation in the terms for acceleration

# Pressure as vertical co- ordinate



$$\frac{1}{\rho} \left( \frac{\partial p}{\partial x} \right)_{t,y,z} \equiv \frac{1}{\rho} \lim_{\delta x \rightarrow 0} \frac{\delta p}{\delta x} = -\frac{1}{\rho} \lim_{\delta x \rightarrow 0} \frac{\rho \delta \phi}{\delta x} = - \left( \frac{\partial \phi}{\partial x} \right)_{t,y,p}$$

$$\frac{D}{Dt} \mathbf{v}_h + f \mathbf{k} \wedge \mathbf{v}_h = -\nabla_p \phi$$

# Thermal wind

Difference of winds at two pressure levels is Thermal Wind  $\mathbf{v}_T$

$$\begin{aligned}\mathbf{v}_T &= \mathbf{v}_{g2} - \mathbf{v}_{g1} = \frac{g}{f} \mathbf{k} \wedge (\nabla_p z_2 - \nabla_p z_1) \\ &= \frac{g}{f} \mathbf{k} \wedge \nabla_p z'\end{aligned}$$

Integrated hydrostatic  
equation gives

$$z' = \frac{R\bar{T}}{g} \ln \frac{p_1}{p_2}$$

$$\mathbf{v}_T = \frac{R}{f} \left( \ln \frac{p_1}{p_2} \right) \mathbf{k} \wedge \nabla \bar{T}$$

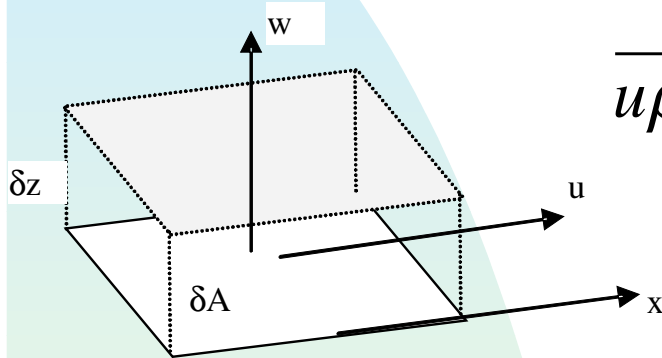
# Eddy friction

Mean

$$\bar{s} = \frac{1}{2Q} \int_{t_0-Q}^{t_0+Q} s dt$$

eddy

$$s' \equiv s - \bar{s}$$



$$\begin{aligned} \overline{u\rho w} &= \rho \overline{(\bar{u} + u')(\bar{w} + w')} = \rho \overline{(\bar{u}\bar{w} + u'\bar{w} + \bar{u}w' + u'w')} \\ &= \rho(\bar{u}\bar{w}) + \rho(\overline{u'w'}) \end{aligned}$$

Upward flux of x-mom<sup>t</sup> due to the eddies =

$$\tau_{xz} = \rho(\overline{u'w'})$$

Accel due to the eddies =

$$-\frac{1}{\rho} \frac{\partial \tau}{\partial z}$$

Introduce eddy diffusion coeff.

$$\tau_{xz} = -\rho K \frac{\partial \bar{u}}{\partial z}$$

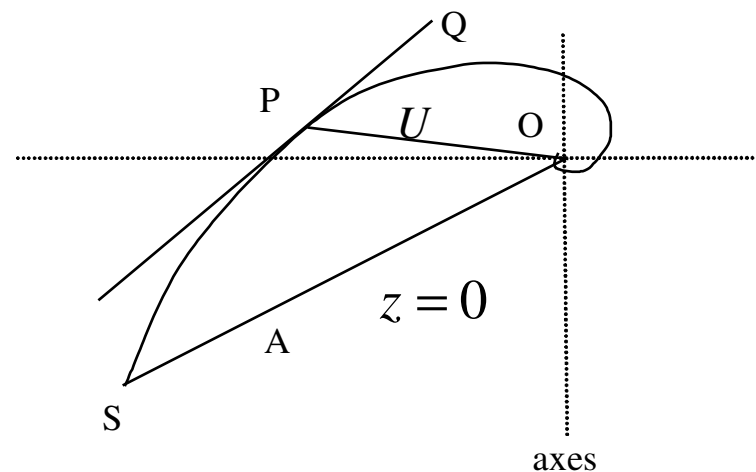
# Ekman Spiral

- Steady unaccelerated flow
- Uniform geostrophic wind in vertical and horizontal
- Variations of density small in bottom km

$$f\mathbf{k} \wedge \bar{\mathbf{v}}_h = -\frac{1}{\rho} \nabla_h p + K \left( \frac{\partial^2 \bar{\mathbf{v}}_h}{\partial z^2} \right)$$

$$f\mathbf{k} \wedge \mathbf{v}_a = +K \left( \frac{\partial^2 \mathbf{v}_a}{\partial z^2} \right)$$

Shape of ageostrophic component



# Vorticity Equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - fv = -\frac{1}{\rho} \frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + fu = -\frac{1}{\rho} \frac{\partial p}{\partial y}$$

$$\left( \frac{D}{Dt} \right)_h (\zeta_{rel} + f) = -(\zeta_{rel} + f) \text{div}_h \mathbf{v}_h$$

$$\left( \frac{D}{Dt} \right)_h (\zeta_{abs}) = -(\zeta_{abs}) \text{div}_h \mathbf{v}$$

$$\zeta_{rel} = \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

$$\zeta_{abs} = \zeta_{rel} + f$$

# Potential vorticity

$$\left( \frac{D}{Dt} \right)_p (\zeta_{abs}) = \zeta_{abs} \frac{\partial \bar{\omega}}{\partial p}$$

$$\frac{1}{\delta p} \frac{D \delta p}{Dt} = \frac{\partial \bar{\omega}}{\partial p}$$

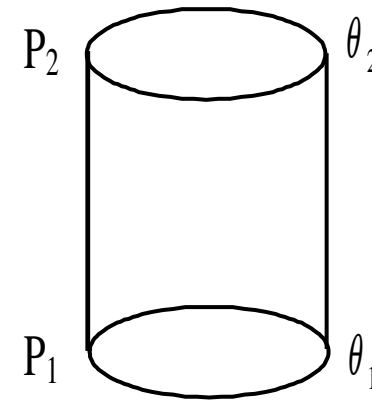
$$\left( \frac{D}{Dt} \right)_p (\zeta_{abs}) - \frac{\zeta_{abs}}{\delta p} \frac{D \delta p}{Dt} = 0$$

$$\left( \frac{D}{Dt} \right)_p \frac{\zeta_{abs}}{\delta p} = 0$$

$$\frac{\zeta_{abs}}{\delta p}$$

**a.k.a. the potential vorticity is conserved  
In frictionless flow**

Atmospheric Dynamics



$$\delta p = p_1 - p_2$$

# Flow over mountain

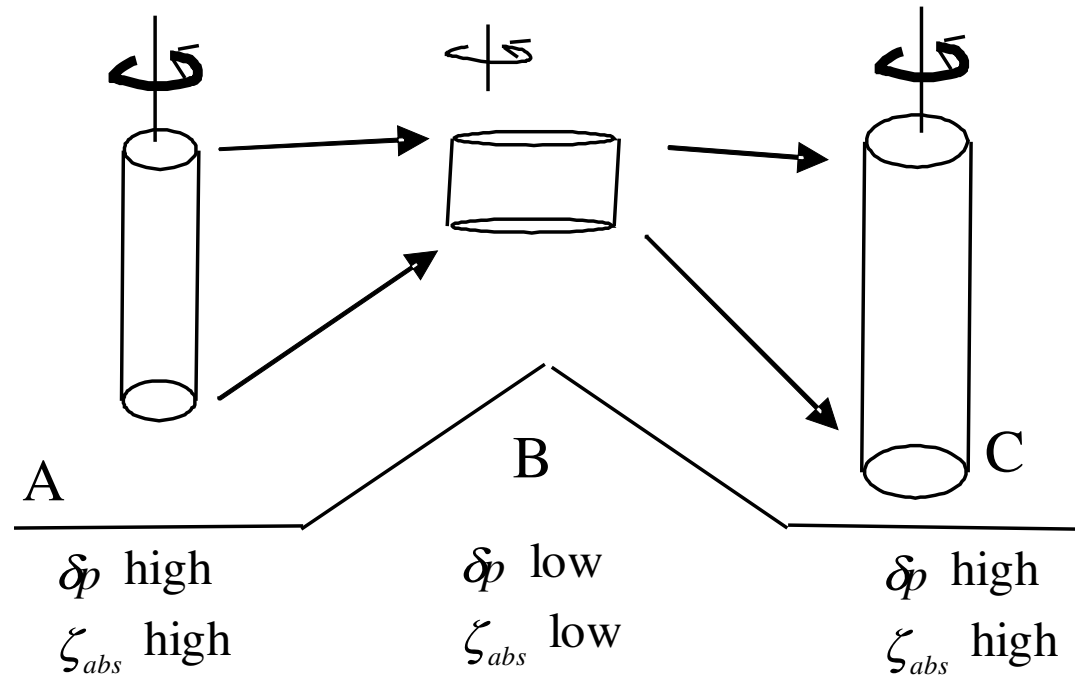
At A  $v=0$

$$\frac{\partial u}{\partial y} = 0$$

$$\zeta_{rel} = 0$$

$$\zeta_{abs} = f$$

At B  $v=0$   $\delta p$  small



$$\zeta_{abs} < f$$

$$\zeta_{rel} < 0$$

$$\frac{\partial v}{\partial x} < 0$$

# f-plane and $\beta$ -plane approximations

- Both use tangent plane approx.
- For f-plane,  $f = \text{constant} = f_0$
- For  $\beta$ -plane,  $f$  is made a linear function of  $y$

$$f = f_0 + \beta(y - y_0) \quad \beta = \left( \frac{\partial f}{a \partial \phi} \right)_0 = \frac{2\Omega \cos \phi_0}{a}$$

for  $\phi_0 = 45^\circ N$

$$\beta = \frac{10.3 \times 10^{-5} s^{-1}}{6371 km} = 1.62 \times 10^{-11} m^{-1} s^{-1}$$

# Stream function

$$u_g = -\frac{\partial \psi}{\partial y} \quad v_g = +\frac{\partial \psi}{\partial x} \quad \longrightarrow \quad \mathbf{v}_g = \mathbf{k} \wedge \nabla_p \psi$$

$$\zeta_{rel} \sim \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla_p^2 \psi$$

$\psi$  is the *streamfunction*

# The Quasi-geostrophic equations

The complete evolution of the flow is governed by the two equations

$$\left( \frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y} \right)_p \nabla_p^2 \psi + v_g \beta = f_0 \frac{\partial \varpi}{\partial p}$$

$$\left( \frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y} \right)_p \frac{\partial \psi}{\partial p} + \frac{\Gamma_0 R}{f_0 p} \varpi = 0$$