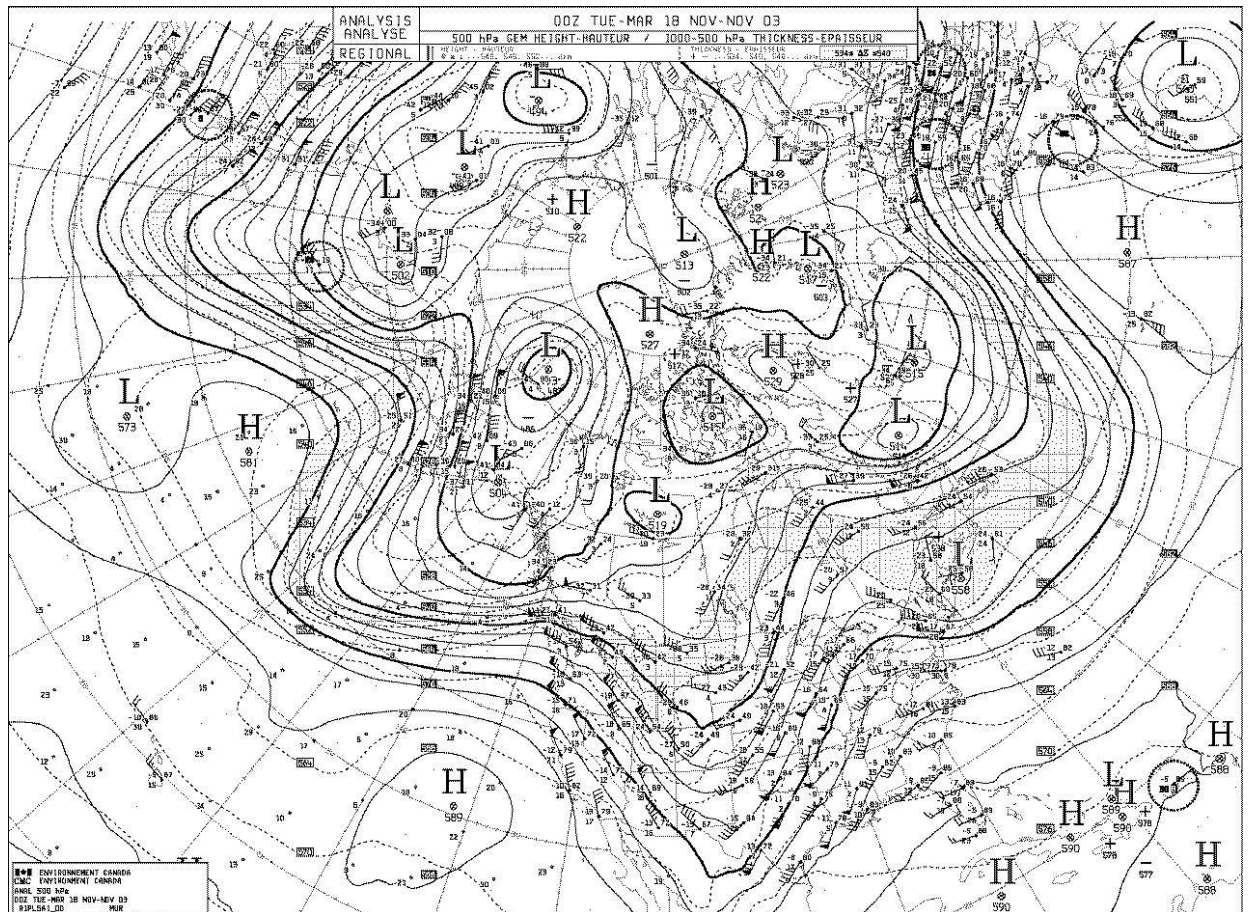


Chapter 14: The Barotropic Rossby wave

The observed flow

The figure shows a typical distribution of winds and pressure at 500hPa in middle latitudes in winter months

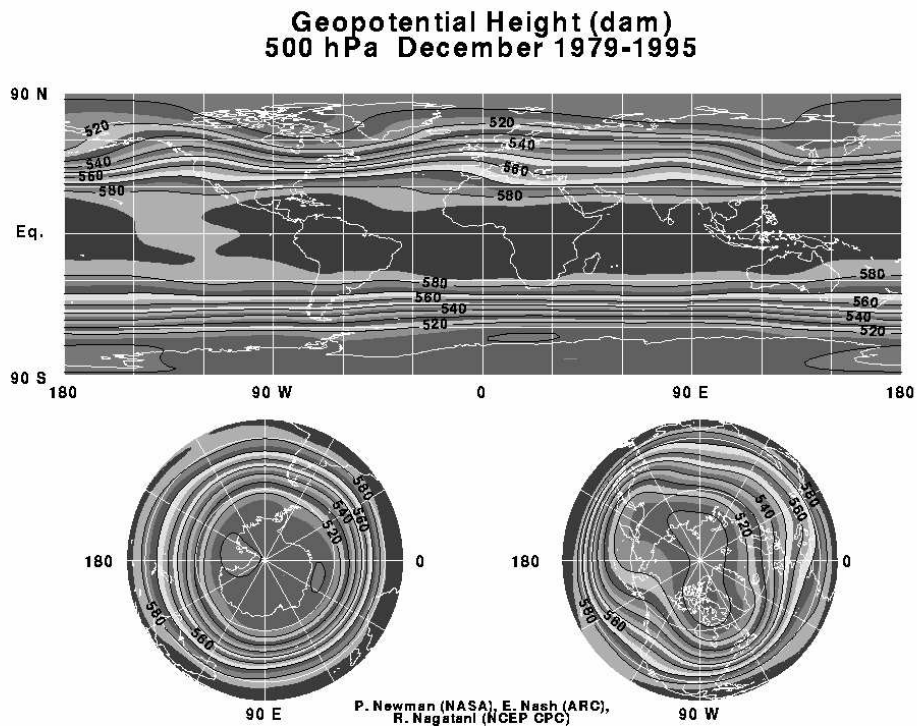


We observe that it is basically westerly but with north-south wave-like meanders. We have a rudimentary understanding of which it is basically westerly. In the first place, the south-north temperature gradient is consistent with a westerly wind component which increases with height, so we can expect to have positive components from west to east once we are sufficiently far from the surface. Moreover the general considerations of angular momentum would lead us to expect westerly winds once we are sufficiently far from the equator. Admittedly those arguments become weak once the flow is not axi-symmetric, but it remains true that air moving away from the equator will have a tendency to develop a westerly drift due to the Coriolis force.

The waves are usually observed to travel from west to east, but not so quickly as the air blows through them. There are typically between about 4 and 6 waves round a latitude circle at these heights.

When the time average is taken, the travelling waves average out to a large extent to give a more zonal flow, but some waviness remains (see the figure below).

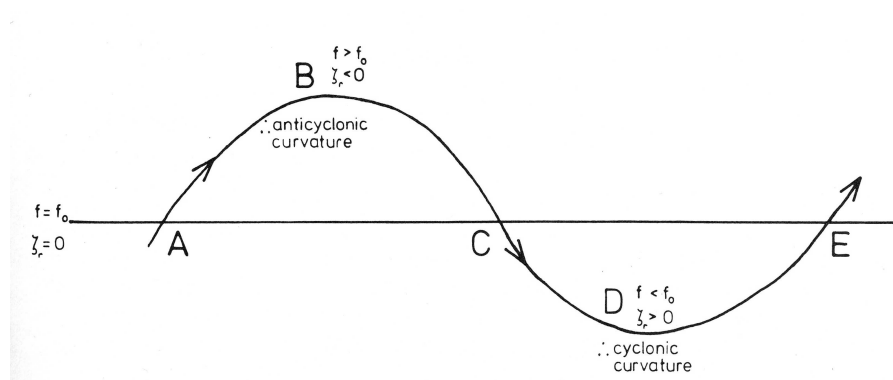
Presumably on the instantaneous map can be regarded as comprised of a mixture of some steady stationary waves and some varying and travelling waves, so that when



the average is taken, only the steady waves remain. There are about 3 peaks or troughs around the latitude circle in the steady, stationary waves. We may ask how these meanders are maintained, or why they exist.

Parcel considerations of wave motion

A first concept of why we might expect wave motion can be obtained by considering the path of an air parcel.



[[Remainder of this section still to be written]]

Barotropic Flow

A fluid is said to be *barotropic* when the density is a function of pressure alone. (Barotropic is Greek for “pressure following”. It is the density which follows the pressure. The atmosphere, strictly speaking, is not a barotropic fluid, but we can

imagine barotropic states. If the atmospheric state is barotropic, then on a surface of constant pressure the density would be constant, and, by the gas law, the temperature would be constant too. Now if the temperature is constant on a constant pressure surface, there can be no thermal wind. It turns out that a barotropic state must have the same wind at all heights. Moreover if the state remains barotropic there can be no vertical velocity either, since a vertical velocity would bring air of different potential temperature onto the pressure surface and disturb the barotropy. Thus the phrase “barotropic flow” has come to be shorthand for horizontal flow which does not change with height.

Observations show that the flow in middle latitudes of the mid-troposphere is basically a westerly flow which meanders north and south in a sort of giant waviness. The first steps at understanding this observed motion can come from a consideration of barotropic flow. Since there is no vertical velocity, the right hand side of the vorticity equation (eq 4, chapter 13) becomes zero, and in terms of the stream-function it becomes

$$\left(\frac{\partial}{\partial t} - \frac{\partial \psi}{\partial y} \frac{\partial}{\partial x} + \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y} \right) \nabla_h^2 \psi + \frac{\partial \psi}{\partial x} \beta = 0.$$

This equation is non-linear in ψ , so we still cannot solve it. Again, as our aim is to seek insight, we can at least begin by looking for situations in which we *can* solve it. A technique which has proven successful in many different physical investigations is to find some simple state which we can solve for, and then to perturb that state by only a small amount. We then keep only first order small terms in our equation and hence obtain a linear equation which we can solve. The solution remains valid just as long as the perturbation remains small.

The simplest basic flow we can have (apart from a state of rest) is a uniform westerly flow. Thus we shall split the geostrophic velocity into two parts thus

$$\mathbf{v}_g = \begin{matrix} (u_0, 0) & + & (u', v') \\ \text{large} & & \text{small} \\ \text{basic} & & \text{perturbation} \\ \text{flow} & & \end{matrix}$$

Note that the subscript zero and the prime do not necessarily have the same meaning as elsewhere in these notes.

We shall likewise split the stream function into the part corresponding to the basic flow and the perturbation part. $\psi = \psi_0 + \psi'$, with $u_0 = -\frac{\partial \psi_0}{\partial y}$ and

$$(u', v') = \left(-\frac{\partial \psi'}{\partial y}, \frac{\partial \psi'}{\partial x} \right).$$

$$\left(\frac{\partial}{\partial t} + (u_0 + u') \frac{\partial}{\partial x} + v' \frac{\partial}{\partial y} \right) \nabla_h^2 (\psi_0 + \psi') + \frac{\partial (\psi_0 + \psi')}{\partial x} \beta = 0$$

$$\left(\frac{\partial}{\partial t} + (u_0 + u') \frac{\partial}{\partial x} + v' \frac{\partial}{\partial y} \right) \nabla_h^2 \psi' + \frac{\partial \psi'}{\partial x} \beta = 0$$

Now we apply the idea that the perturbation quantities are small compared to the basic flow quantities. This means that terms involving squares of perturbation quantities can be neglected in comparison with terms in single powers. This gives

$$\left(\frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x} \right) \nabla_h^2 \psi' + \frac{\partial \psi'}{\partial x} \beta = 0.$$

Eq 1

This is now a linear partial differential equation for the stream function. Furthermore it has constant coefficients, so it is easy to solve by standard techniques. It may be subject to boundary conditions or initial conditions which we have not yet specified.

The general solution is will be a linear combination of terms of the form

$$\psi' = A \exp i\{\sigma t + \lambda x + \mu y\}.$$

Eq 2

Where A, σ, λ, μ are constant.

The solutions we get are wave-like, as we shall see in more detail below. The waves in the solution are known as barotropic Rossby waves after Rossby who first studied them.

An aside: reminder on wave properties

We note that when t increases by $2\pi/\sigma$ while x and y are held constant ψ' repeats. Hence the motion is periodic, with period $2\pi/\sigma$ and frequency $\sigma/2\pi$. Likewise the motion repeats when x increases by $2\pi/\lambda$ with t and y held constant and when y increases by $2\pi/\mu$ with t and x held constant. Thus provided λ and μ are real the motion is periodic in space, that is to say, it is wavelike in space. λ and μ are said to be *wavenumbers* in the x and y directions, though the name is a little unfortunate, as they are dimensional quantities having dimensions $(\text{length})^{-1}$. The wavelengths in those directions are

$$L_x = 2\pi/\lambda \text{ and } L_y = 2\pi/\mu$$

respectively. A is the amplitude, and at this stage it might be real or imaginary.

As just stated, the solution will be a linear combination of terms of that form. To get a real solution we need to add together several things like the right side of the equation. We can get a real quantity by adding or subtracting two components with the same amplitude and with frequency and wavenumbers which are of opposite signs. Thus

$$A \exp i\{\sigma t + \lambda x + \mu y\} + A \exp -i\{\sigma t + \lambda x + \mu y\} = 2A \cos\{\sigma t + \lambda x + \mu y\}$$

while subtracting two components would give a real solution involving the sine function. Further adding two solutions one in terms of the cosine and one in terms of the sine with different amplitude but with the same values period and wavelength leads to an expression which can be written in the form $B \cos\{\sigma + \lambda(x - \varepsilon) + \mu y\}$.

ε is called the *phase* of the wave, but the usage is a bit slack and $\{\sigma + \lambda(x - \varepsilon) + \mu y\}$ is sometimes called the phase of the wave also.

For a solution written in the form $B \cos\{\sigma + \lambda(x - \varepsilon) + \mu y\}$, the wave will have maxima where $\{\sigma + \lambda(x - \varepsilon) + \mu y\} = 0 \pm 2\pi n$ and minima where $\{\sigma + \lambda(x - \varepsilon) + \mu y\} = \pi \pm 2\pi n$. These can be regarded as lines in the x-y plane, sometimes called *phaselines*. You should familiarise yourself with which way these lines slope for various combinations of positive and negative values of the wavenumbers.

Finally we note that we could also write $\{\sigma + \lambda(x - \varepsilon) + \mu y\}$ in the form $\{\lambda(x - ct - \varepsilon) + \mu y\}$ where

$$c = -\sigma / \lambda.$$

This makes it clear that as t increases the phase lines can be found further east by an amount ct . Hence c is the speed that the wave moves in the x-direction.

The nature of the solution

We note that $\frac{\partial}{\partial t} \psi' = i\sigma A \exp i\{\sigma + \lambda x + \mu y\} = i\sigma \psi'$ and likewise that $\frac{\partial}{\partial x} \psi' = i\lambda \psi'$

and $\frac{\partial}{\partial y} \psi' = i\mu \psi'$, so that Eq 1 becomes $(i\sigma + iu_0\lambda)(-\lambda^2 - \mu^2) + i\beta\lambda = 0$ which can

be re-arranged to

$$c = -\frac{\sigma}{\lambda} = u_0 - \frac{\beta}{\lambda^2 + \mu^2}$$

Eq 3

This is the dispersion relation for the waves. It describes how the frequency (and hence phase-speed) is related to the wavenumbers (and hence wavelengths) of the waves if they are to be solutions to Eq 1.

The denominator of the final term in Eq 3 is always positive. We deduce therefore from the minus sign which precedes the final term that the waves move eastward less quickly than the basic flow, or stated differently the waves move westward relative to the basic flow. Note that the final term will be large for small wave numbers (large wavelengths) and will be small for small wavelengths. (Small wavelengths “blow with the basic wind”).

We have not considered how the waves are generated; why they might be there. One mechanism for producing them is the deflection of air by mountain waves which we saw in the previous chapter. This initiates a southward movement in the airflow. As

the mountains are stationary, for steady values of u_0 the waves must stay in the same place. Thus we expect waves generated by airflow over mountains to have $c = 0$. This happens when

$$\lambda^2 + \mu^2 = \frac{\beta}{u_0}$$

Consider waves with $\frac{\partial}{\partial y} = 0$. Clearly the phaselines lie north-south for such waves and $\mu = 0$. Hence

$$\left(\frac{2\pi}{L_x}\right)^2 = \frac{\beta}{u_0}$$

or

$$L_x = 2\pi \sqrt{\frac{u_0}{\beta}}.$$

Putting $\beta = 1.11 \times 10^{-11} m^{-1} s^{-1}$ which is the value for 60°N. gives the following relation between basic windspeed and wavelength.

| | | | |
|-----------------|------|------|------|
| u_0 / ms^{-1} | 10 | 15 | 20 |
| Wavelength / km | 5960 | 7300 | 8430 |

Thus according to the strength of the basic wind there is the possibility of having 2, 3, or 4 waves around a latitude circle. As we did the analysis on a beta-plane rather than a sphere we should not expect the wavelength to be an exact integral fraction of the length of a latitude circle. Note however that we are not constrained actually to a particular value of the x-wavenumber for stationary waves with a given basic wind because we can vary the y-wavenumber too.