Atmospheric Dynamics

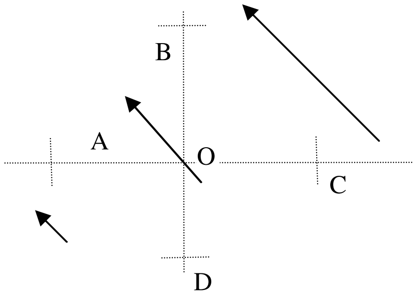
Solutions to problems on Chapters 10

1) At surface resolving the wind into components, we get $(u,v) = (14.14\cos 45, 14.14\cos 45)ms^{-1} = (10,10)ms^{-1}$. While at height of 4 km, $(u,v) = (10,-10)ms^{-1}$

Now when $w = 0$, x-component of vorticity is $\zeta_x = -\frac{\partial v}{\partial z} \sim -\frac{(10ms^{-1}) - (-10ms^{-1})}{4km} = -\frac{20}{4000}s^{-1}$. The other component is found in a similar manner.. Giving

$\zeta_x = -5\times 10^{-3}s^{-1}$ and $\zeta_y = 0s^{-1}$

2) There is no one single way of tackling this problem, so this is only one possible approach.



Set up a grid of points around O as shown. These have been labelled ABCD in the sketch. The plan is to work out u, v at these points and then use the values to estimate $\frac{\partial v}{\partial x}$ and $\frac{\partial u}{\partial y}$. The size of the grid can be arbitrarily chosen, but it will be convenient to make OA=OB=OC=OD=100km.

At C (and at B) the windspeed is $(20ms^{-1}/500km)\times(100km/\sqrt{2})$ more than at O. So v is $(20ms^{-1}/500km)\times(100km/\sqrt{2})/\sqrt{2}$ greater at C than at O, giving that $\frac{\partial v}{\partial x} = \frac{2ms^{-1}}{100km} = 2\times 10^{-5}s^{-1}$.

A similar argument shows that $\frac{\partial u}{\partial u} = -2\times 10^{-5}s^{-1}$. So that $\boxed{\zeta_{rel} = 4\times 10^{-5}s^{-1}}$.

At 45°N $f = 10.3\times 10^{-5}s^{-1}$, so $\boxed{\zeta_{abs} = 14.3\times 10^{-5}s^{-1}}$

An alternative approach to this, which slightly simplifies the arithmetic, is to note that the vorticity is twice the local spin of the fluid particle. This means that vorticity must be invariant under a rotation of the axes, so you could take a new set of axes, at 45 degrees to the original ones. If this is done with the new x-axis to the NE, then only $\frac{\partial v}{\partial x}$ needs to be computed in these new axes, as u is zero in them. What is more the gradient is then trivial to compute too.

3) The general idea of how to tackle this problem will probably suggested to you from the way we obtained the interpretation of the vorticity. However, there are several ways of going about this, and I show only one. The diagram is rather time-consuming to draw in this word processing package, so I hope a description will be adequate for now.

Consider a "lump" of fluid which is initially a square with edges parallel to the axes. After time δt this has moved to a new position, the edges will have changed length slightly and the edges will have slightly different lengths. Let the lengths of the edges be $\delta x, \delta y$. The area A is given to first order by $\delta x . \delta y$. (Note that this is true even if the edges have become slightly non-parallel with the axes, as any error in that arises from assuming that the cosine of the angle made with the axes differs from 1, and the difference of that cosine from 1 is second order in the angle).

Now $\frac{1}{\delta A}\frac{D}{Dt}\delta A = \frac{D}{Dt}\ln A = \frac{D}{Dt}\ln(\delta x.\delta y) = \frac{1}{\delta x}\frac{D}{Dt}\delta x + \frac{1}{\delta y}\frac{D}{Dt}\delta y$

Eq 1

Think now of the edge of the area which is initially parallel to the x-axis and has endpoints P and Q. Suppose P is the point (x_0, y_0) and Q is $(x_0 + \delta x, y_0)$. In time δt P will move to P′ and Q to Q′, say. It is easy to show that the x-co-ordinate of P′ is point $x_0 + u\delta t$, and that the x-co-ordinate of Q′ is $x_0 + \left\{u + \frac{\partial u}{\partial x}\delta x\right\}\delta t$. Hence the change in δx in time δt is $\frac{\partial u}{\partial x}\delta x \delta t$. It follows that $\frac{1}{\delta x}\frac{D\delta x}{Dt} = \frac{\partial u}{\partial x}$.

The changes to the length parallel to the y-axis can be treated similarly so that from Eq 1
$\frac{1}{\delta A}\frac{D}{Dt}\delta A = \frac{1}{\delta x}\frac{D}{Dt}\delta x + \frac{1}{\delta y}\frac{D}{Dt}\delta y = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$

The right hand side is what we have defined as $div_h \mathbf{v}_h$, so the result is proved.

4) In components we see that $\mathbf{v}_r = \left(-\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x}\right)$, so that for $\mathbf{v}_r$ alone $\zeta_{rel} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}$. This is the Laplacian of ψ. There is no reason which it should be zero.

On the other hand $div_h v_r = -\frac{\partial}{\partial x}\frac{\partial \psi}{\partial y} + \frac{\partial}{\partial y}\frac{\partial \psi}{\partial x} = 0$. Thus $\mathbf{v}_r$ has rotation, but is non-divergent.

The proof that $\mathbf{v}_d$ has divergence but not vorticity is entirely analogous.

To find ψ for a general wind field, we first compute the vorticity, then ψ is found by solving $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \zeta_{rel}$ subject to suitable boundary conditions.

To find Φ for a general wind field, we first compute the divergence, then Φ is found by solving $\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = div_h v_h$ subject to suitable boundary conditions.