

Atmospheric Dynamics

Lect.10: Vorticity and Divergence

Stephan Matthiesen

Institute for Atmospheric and Environmental Sciences

School of GeoSciences, Edinburgh University

Crew Building room 218

Scope of Lecture

- Definition of vorticity and divergence
- Physical interpretation
- Approximate non-divergence of synoptic-scale motions

Definition of vorticity

3-D vorticity is defined as the curl of the wind velocity

$$\mathit{curl}\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \mathbf{i} \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) + \mathbf{j} \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) + \mathbf{k} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

Since vertical scales and velocities are small we have

$$\mathit{curl}\mathbf{v} \sim -\mathbf{i} \left(\frac{\partial v}{\partial z} \right) + \mathbf{j} \left(\frac{\partial u}{\partial z} \right) + \mathbf{k} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

Magnitude of components

Horizontal components $\sim (10\text{ms}^{-1})/(10\text{km})\sim 10^{-3}\text{s}^{-1}$

Vertical component $\sim (10\text{ms}^{-1})/(1000\text{km})\sim 10^{-5}\text{s}^{-1}$

So horizontal components are about 100 times the vertical component.

Relative vorticity

As velocity components are relative to Earth a subscript *rel* is applied.

Vertical component is of most meteorological relevance despite size.

Often just called *the vorticity*

or *the relative vorticity*

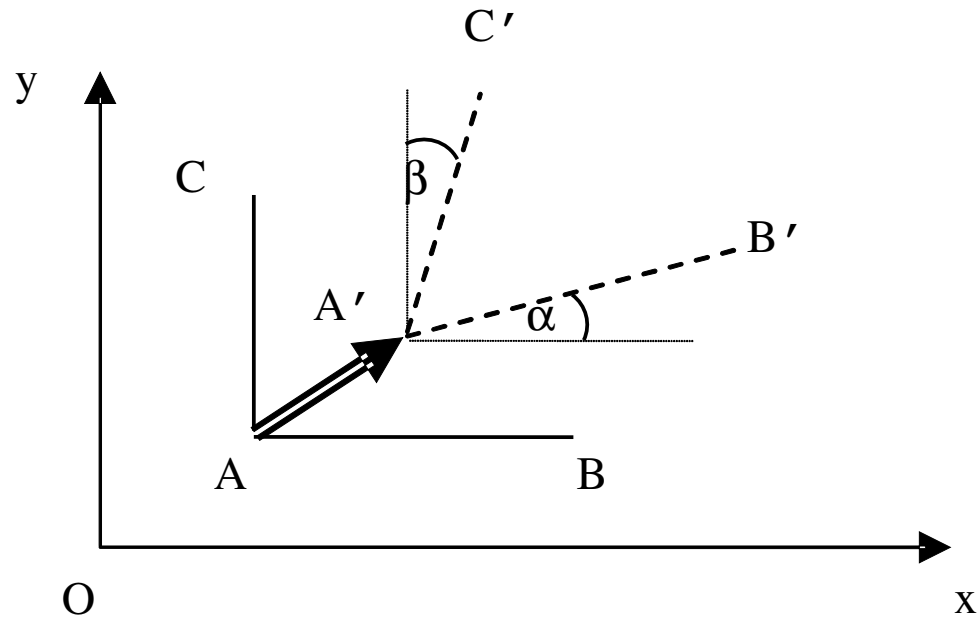
$$\zeta_{rel} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

Physical interpretation of vorticity

Marked lines of fluid AB, AC move to A'B' and A'C' in time δt
 A, B, C are

(x_o, y_o) $(x_o + \delta x, y_o)$ $(x_o, y_o + \delta y)$ **respectively**

A' is easily seen to be $(x_o + u_o \delta t, y_o + v_o \delta t)$



Physical interpretation of vorticity (cont)

A' is

$$(x_o + u_o \delta t, y_o + v_o \delta t)$$

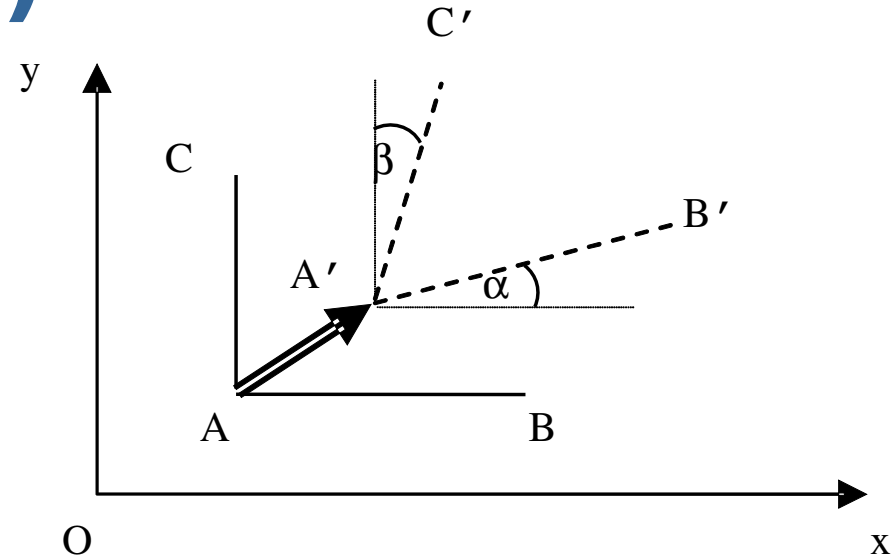
B' is

$$\left(x_o + \delta x + \left(u_o + \frac{\partial u_o}{\partial x} \delta x \right) \delta t, y_o + \left(v_o + \frac{\partial v_o}{\partial x} \delta x \right) \delta t \right)$$

x-comp. of A'B' = $\delta x + O(\delta x \delta t)$

y-comp. of A'B' = $\frac{\partial v}{\partial x} \delta x \delta t + O(\delta x^2 \delta t)$

--> $\alpha = \frac{\partial v}{\partial x} \delta t$

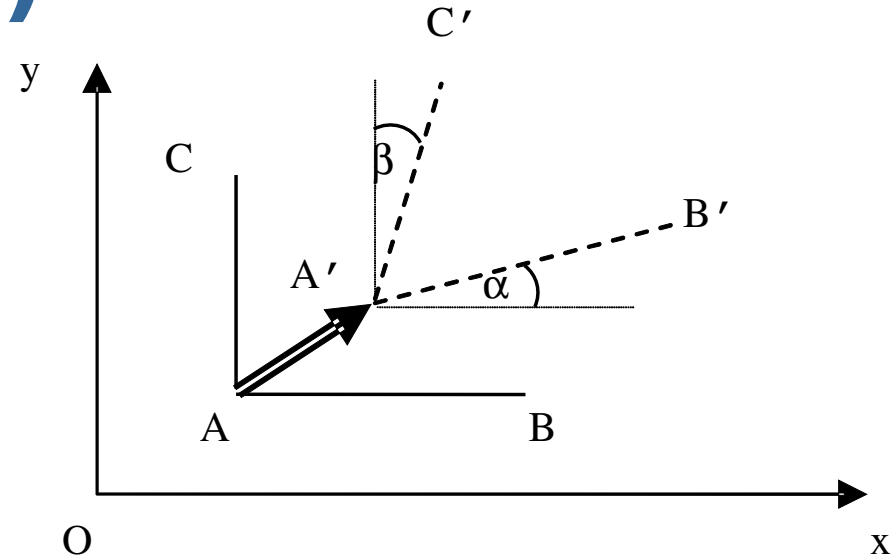


Physical interpretation of vorticity (cont)

$$\alpha = \frac{\partial v}{\partial x} \delta t$$

likewise

$$\beta = \frac{\partial u}{\partial y} \delta t$$



Average rotation rate $= \frac{1}{2}(\alpha - \beta) / \delta t$

Vorticity=twice rate of rotation.

Absolute vorticity

To get absolute vorticity add on twice earth's rotation about vertical

$$\zeta_{abs} = \zeta_{rel} + 2 \times \Omega \sin \phi$$

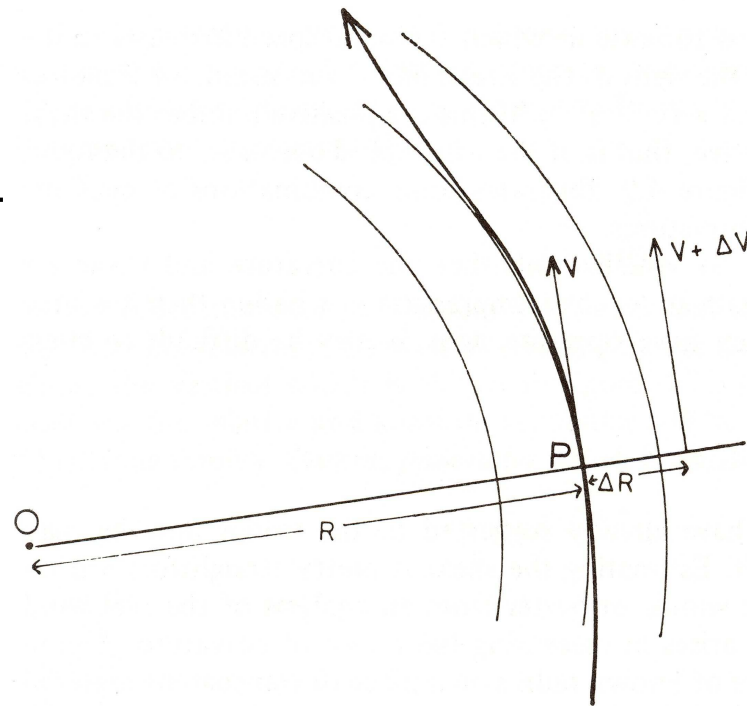
$$\zeta_{abs} = \zeta_{rel} + f$$

Alternative form

- From definition vorticity must be independent of the axes
- Take local axes $Ox'y'$ with y' axis in direction of flow and x' perpendicular to it. So for anticlockwise flow (i.e. cyclonic in N. Hemis.) is radially outwards away from local centre of curvature

$$\zeta = \frac{\partial v'}{\partial x'} - \frac{\partial u'}{\partial y'} = \frac{\partial V}{\partial R} + \frac{V}{R}$$

where R is
the radius of
curvature



Atm

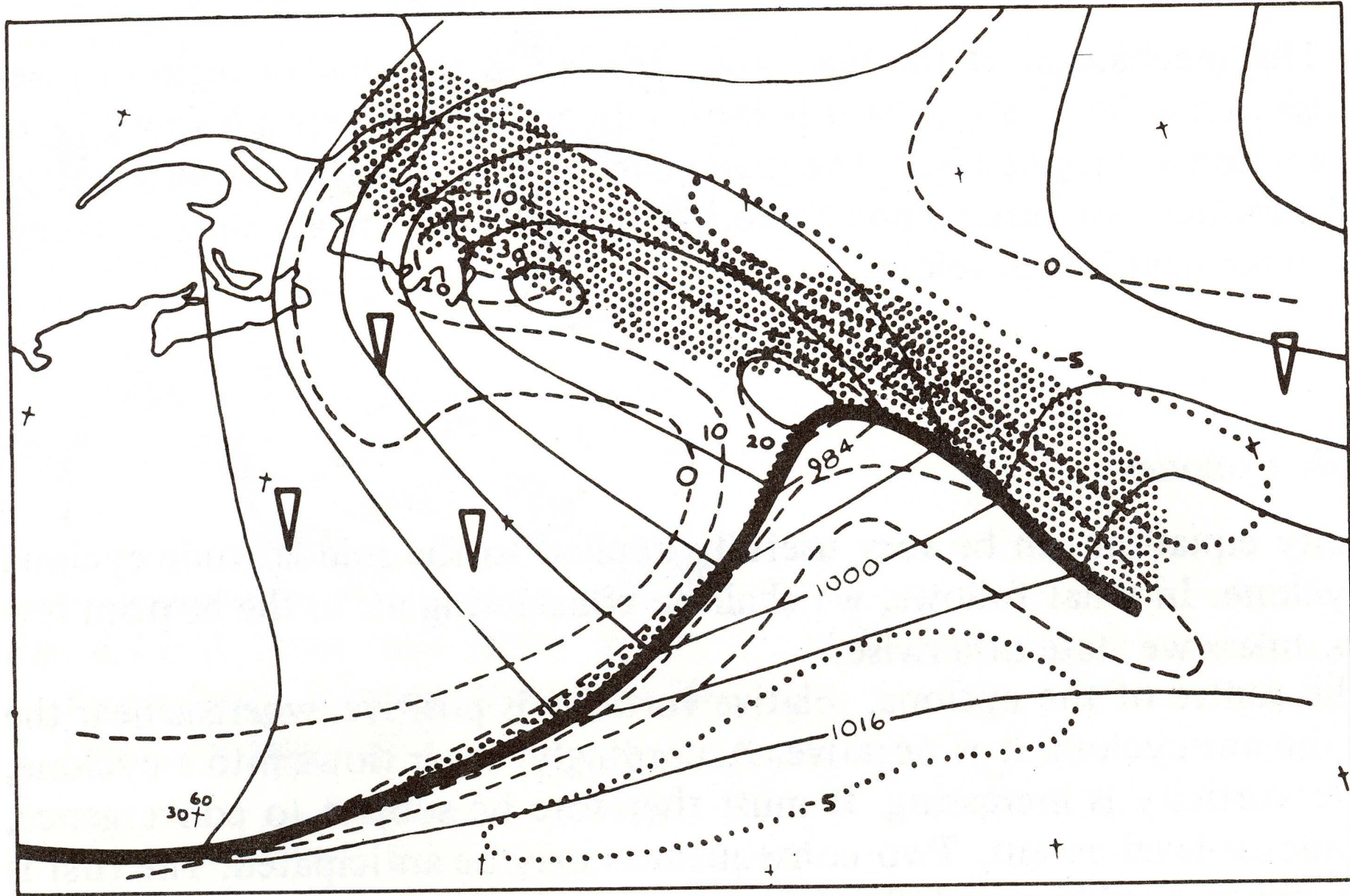
Vorticity

$$\zeta = \frac{\partial V}{\partial R} + \frac{V}{R}$$

Net vertical vorticity is sum of two parts:

- Shear vorticity
- Curvature vorticity

Surface vorticity (estimated from geostrophic wind)



Dashed lines are lines of equal rel. vorticity: units of 10^{-5}s^{-1}

Divergence

3D divergence defined:-

$$\mathit{div}\mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

2D version most useful

$$\mathit{div}_h \mathbf{v}_h = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

We can show that for element δA ,

$$\mathit{div}_h \mathbf{v}_h = \frac{1}{\delta A} \frac{D\delta A}{Dt}$$

**Parcels expand for +ve divergence
contract for -ve divergence**

= convergence

Divergence is small

Consider div of geostrophic wind

$$\begin{aligned} \operatorname{div}_h \mathbf{v}_g &= \left(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} \right) = -\frac{\partial}{\partial x} \left(\frac{1}{\rho f} \frac{\partial p}{\partial y} \right) + \frac{\partial}{\partial y} \left(\frac{1}{\rho f} \frac{\partial p}{\partial x} \right) \\ &= \frac{1}{\rho f} \left[-\frac{\partial^2 p}{\partial x \partial y} + \frac{\partial^2 p}{\partial y \partial x} \right] - \frac{1}{\rho f^2} \frac{\partial p}{\partial x} \frac{\partial f}{\partial y} \end{aligned}$$

hence $\operatorname{div}_h v_g = 0 - \frac{v_g}{f} \frac{\partial f}{\partial y}$

now $\frac{1}{f} \frac{\partial f}{\partial y} = \frac{1}{2\Omega \sin \phi} \frac{1}{a} \frac{\partial}{\partial \phi} 2\Omega \sin \phi = \frac{1}{a} \cot \phi$

Smallness of divergence (cont)

At 45°N $\frac{1}{a} \cot \phi = \frac{1}{6.4 \times 10^6 m} = 10^{-7} m^{-1}$

$$\text{div}_h \mathbf{v}_g \sim 10 \text{ms}^{-1} 10^{-7} m^{-1} = 10^{-6} s^{-1}$$

Thus the divergence is about 10% of the magnitude of the terms which comprise it. These are therefore approximately equal and of opposite sign

As \mathbf{v}_g is an $O(R_0)$ approx to \mathbf{v} , real wind is quasi-nondivergent too